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On the Goeritz matrix of links

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§ 1. Introduction

Let L be an unoriented link in R^3 and $D(L)$ a link diagram of L on R^2. First of all, we consider the case that $D(L)$ is connected. Then R^2 is divided into connected regions by $D(L)$ which can be colored white and black alternately in such a way that adjacent regions are of distinct colors. For each crossing P of $D(L)$, if the white regions at P are distinct, we assign the incident number $\theta(\mathrm{P})$ with 1 or -1 according as the overcrossing line coincides with the undercrossing line by turning over the black regions at P counterclockwise or clockwise. If the white regions at P are same, then we define $\theta(\mathrm{P})$ to be 0. Let $W_0, W_1, W_2, \ldots, W_m$ be the white regions. For $0 \leq i \leq m$ and $0 \leq j \leq m$, we define g_{ij} as follows:

$$g_{ij}=\begin{cases} \displaystyle\sum_{\mathrm{P}\in\partial W_i} \theta(\mathrm{P}) & \textit{if} \quad i=j \\ \displaystyle -\sum_{\mathrm{P}\in W_i \cap W_j} \theta(\mathrm{P}) & \textit{if} \quad i \neq j \end{cases}$$

Define the $(m+1)$-square matrix $\widetilde{G}(L)$ by $\widetilde{G}(L)=(g_{ij})$. The m-square matrix $G(L)$ obtained from $\widetilde{G}(L)$ by deleting the row and the column corresponding to any white region W_{i_0} is called *the Goeritz matrix associated to* $D(L)$.

Next we consider the case that $D(L)$ is disconnected. In this case, $D(L)$ can be expressed as the union of a finite number of link diagrams $D(L_1), D(L_2), \ldots, D(L_n)$ such that $D(L_1), D(L_2), \ldots, D(L_n)$ are connected and are disjoint each other. We construct $\widetilde{G}(L_1), \widetilde{G}(L_2), \ldots, \widetilde{G}(L_n)$ for link diagrams $D(L_1), D(L_2), \ldots, D(L_n)$, respectively. Then we define the square matrix $\widetilde{G}(L)$ by

$$\widetilde{G}(L)=\begin{pmatrix} \widetilde{G}(L_1) & 0 & \ldots & 0 \\ 0 & \widetilde{G}(L_2) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \widetilde{G}(L_n) \end{pmatrix}$$

The matrix $G(L)$ obtained from $\widetilde{G}(L)$ by deleting the i_0-th row and the i_0-th column for any i_0 is called *the Goeritz matrix associated to* $D(L)$ for this case. In either case, it is easy to see that $G(L)$ is a symmetric matrix.

Now let L be an oriented link and $D(L)$ its diagram. A crossing P of $D(L)$ is said to be *of type* II if the oriented overcrossing line coincides with the oriented undercrossing line by turning over the black regions at P either counterclockwise or clockwise. Otherwise, it is said to be *of type* I. We define $\mu(D(L))=\sum\theta(\mathrm{P})$ (summed over all crossings P of type II).

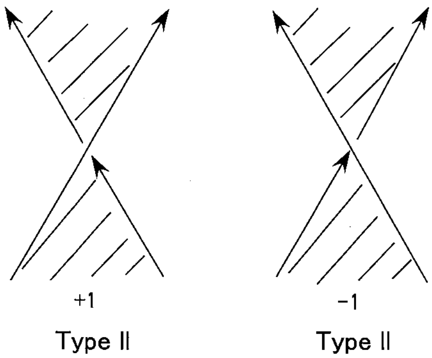


Fig. 1. 1

For two square matrices A and B with integral entries, A *is said to be equivalent to* B if A can be transformed to B by a finite sequence of the following two operations and their inverses:

$$\mathrm{I}: M \to {}^tPMP \quad (P \text{ is an invertible matrix with integral entries})$$

$$\mathrm{II}: M \to \begin{pmatrix} M & 0 \\ 0 & \pm 1 \end{pmatrix}$$

It is well-known (See [3, 6]) that

(1) Two unoriented (oriented) links L and L' are of the same link type if and only if $D(L)$ is transformed to $D(L')$ by a finite sequence of the operations Type Ia, Type Ib, Type II, Type IIIa, Type IIIb (See Fig. 1. 2) and their inverses, which are called the Reidemeister moves.

(2) If L and L' are unoriented links of the same link type, then $G(L)$ is equivalent to $G(L')$.

In section 2, we will give a detailed proof for (2) which will be used later in this paper. Throughout the paper, we define $\#W(D(L))$ to be the number of white regions of $D(L)$. We often denote it simply by $\#W$. Also, for a symmetric matrix A, sign$A$ will denote the signature of A, i.e., when A is diagonalized, the signature of

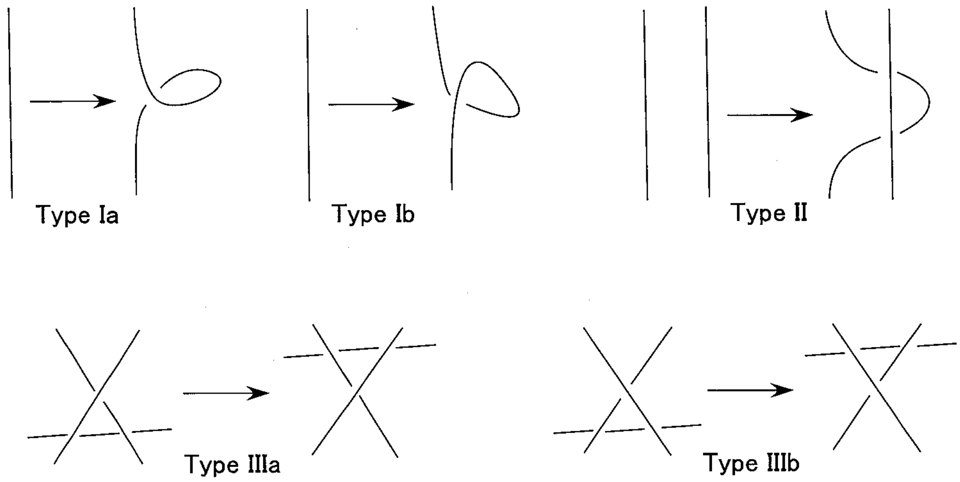


Fig. 1. 2

A is defined to be the number of positive diagonal elements minus the number of negative diagonal elements (See [4, 5]). The purpose of this paper is to prove the following three theorems by using combinatorial methods based on the Reidemeister moves:

THEOREM 1 *Let L be an oriented link and $D(L)$ its diagram. Then* $\#W(D(L))-\mu(D(L))$ (mod 2) *is an invariant of the oriented link type of L.*

THEOREM 2 *Let L be an oriented link and $D(L)$ its diagram. Then* $\mathrm{rank}G(L)-\mu(D(L))$ (mod 2) *is an invariant of the oriented link type of L.*

THEOREM 3 *Let L be an oriented link and $D(L)$ its diagram. Then* $\mathrm{sign}G(L)-\mu(D(L))$ *is an invariant of the oriented link type of L.*

The proof for Theorem 3 given in the paper is an alternate proof for [2] and we will use purely combinatorial technique. $\mathrm{sign}G(L)-\mu(D(L))$ will be denoted by $\sigma(L)$ and is called *the signature of an oriented link L* ([1], [5]).

§ 2. The Reidemeister Moves and the Goeritz matrices

In this section, we examine the behavior of the Goeritz matrices under the Reidemeister moves. We need to consider the following ten cases shown in the figures from Fig. 2. 1 to Fig. 2. 10:

Type Ia (Case 1)

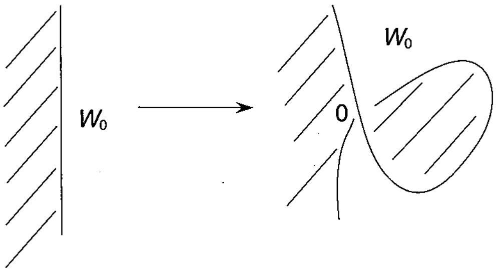


Fig. 2. 1

Type Ia (Case 2)

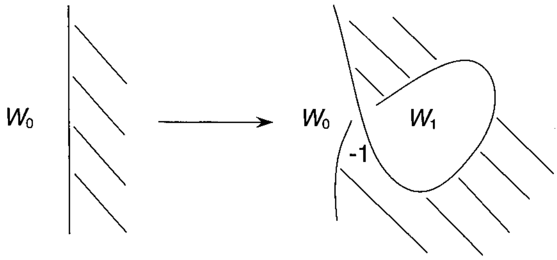


Fig. 2. 2

Type Ib (Case 1)

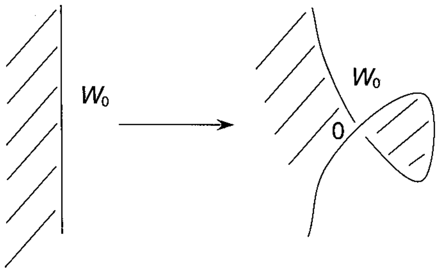


Fig. 2. 3

Type Ib (Case 2)

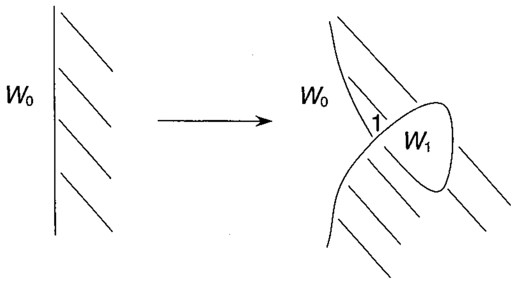


Fig. 2. 4

Type II (Case 1)

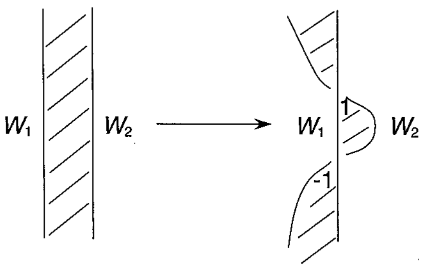


Fig. 2. 5

Type II (Case 2)

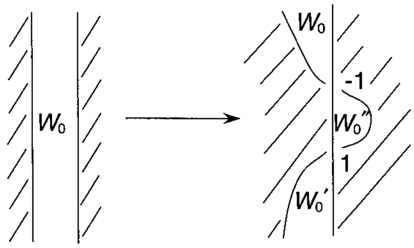


Fig. 2. 6

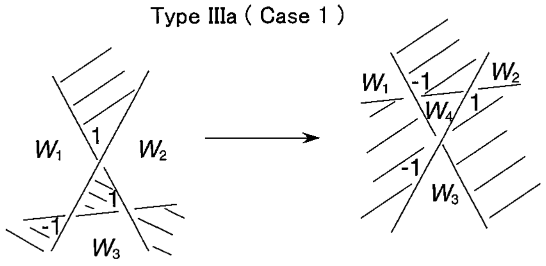


Fig. 2. 7

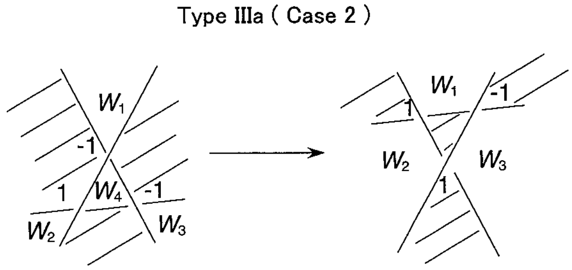


Fig. 2. 8

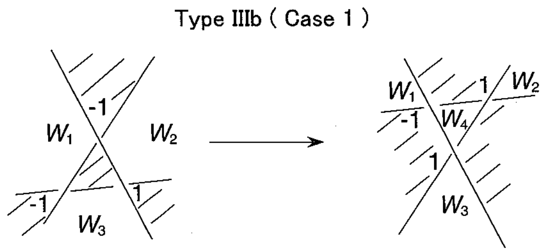


Fig. 2. 9

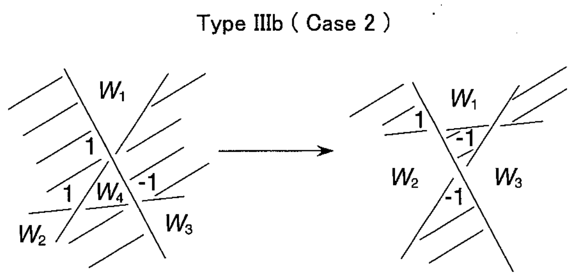


Fig. 2. 10

It is easy to see that in the cases Type Ia(Case 1), Type Ib(Case 1) and Type II (Case 1), the Georitz matrices and $\#W$ are preserved. We note that in Type II(Case 1) the number of components of the link diagram may happen to change. Even in this case, we can show that the Goeritz matrices and $\#W$ are unaltered.

We will consider the remaining seven cases. In the following argument, the matrix $\widetilde{G}(L)$ of the link diagram $D(L)$ will change from $\widetilde{A}$ to $\widetilde{B}$ by the operation shown in each figure. The corresponding Goeritz matrices will be denoted by A and B, respectively. The number of white regions also change from $\#W$ to $\#W'$.

(1) Type Ia (Case 2)

In this case, we have

$$\widetilde{B}=\left[\begin{array}{c|c} & 1 \\ \widetilde{A} & 0 \\ & \vdots \\ & 0 \\ \hline 1\ 0 \cdots 0 & -1 \end{array}\right]$$

In $\widetilde{B}$, the first row and the first column correspond to W_0 and the last row and the last column correspond to W_1. By deleting the first row and the first column of $\widetilde{A}$ and $\widetilde{B}$, we have

$$B=\begin{pmatrix} A & O \\ O & -1 \end{pmatrix},$$

which implies that $\operatorname{rank}B=\operatorname{rank}A+1$, $\#W'=\#W+1$ and $\operatorname{sign}B=\operatorname{sign}A-1$.

(2) Type Ib (Case 2)

In this case, we have the following:

$$\widetilde{B}=\left(\begin{array}{ccc|c} & & & -1 \\ & \widetilde{A} & & 0 \\ & & & \vdots \\ & & & 0 \\ \hline -1 & 0 \cdots & 0 & 1 \end{array}\right)$$

In $\widetilde{B}$, the first row and the first column correspond to W_0 and the last row and the last column correspond to W_1. By deleting the first row and the first column, we have

$$B=\begin{pmatrix} A & O \\ O & 1 \end{pmatrix}$$

Hence it follows that $\operatorname{rank}B=\operatorname{rank}A+1$, $\#W'=\#W+1$ and $\operatorname{sign}B=\operatorname{sign}A+1$.

(3) Type II (Case 2)

First of all we assume that the number of components of the link diagrams is not changed. In this case, we have

$$\widetilde{A}=\left(\begin{array}{c|c} a_{00} & \cdots\ a_{0j}\ \cdots \\ \hline \vdots & \\ a_{j0} & A \\ \vdots & \end{array}\right), \quad \widetilde{B}=\left(\begin{array}{ccc|c} b_{00} & 0 & 1 & \cdots\ b_{0j}\ \cdots \\ 0 & b'_{00} & -1 & \cdots\ b'_{0j}\ \cdots \\ 1 & -1 & 0 & \cdots\ 0\ \cdots \\ \hline \vdots & \vdots & \vdots & \\ b_{j0} & b'_{j0} & 0 & A \\ \vdots & \vdots & \vdots & \end{array}\right)$$

Here, the first row and the first column of $\widetilde{A}$ and $\widetilde{B}$ correspond to W_0. In $\widetilde{B}$, the second row and the second column correspond to W'_0 and the third row and third column to W_0''. Furthermore, $a_{0j}=b_{0j}+b'_{0j}$ holds. By deleting the first row and the first column of $\widetilde{A}$ and $\widetilde{B}$, we have

$$B=\left(\begin{array}{cc|ccc} b'_{00} & -1 & \cdots & b'_{0j} & \cdots \\ -1 & 0 & \cdots & 0 & \cdots \\ \hline \vdots & \vdots & & & \\ b'_{j0} & 0 & & A & \\ \vdots & \vdots & & & \end{array}\right)$$

Repeated applications of the operation I yield

$$\left(\begin{array}{cc|c} b'_{00} & -1 & \multirow{2}{*}{O} \\ -1 & 0 & \\ \hline \multicolumn{2}{c|}{O} & A \end{array}\right)$$

If b'_{00} is an odd number, by using the operation I several times and by adding the first row to the second row and the first column to the second column, we have

$$\left(\begin{array}{cc|c} 1 & -1 & \multirow{2}{*}{O} \\ -1 & 0 & \\ \hline \multicolumn{2}{c|}{O} & A \end{array}\right), \quad \left(\begin{array}{cc|c} 1 & 0 & \multirow{2}{*}{O} \\ 0 & -1 & \\ \hline \multicolumn{2}{c|}{O} & A \end{array}\right)$$

If b'_{00} is an even number, by applying the operation I several times and by multiplying the first row by -1 and the first column by -1, we obtain

$$\left(\begin{array}{cc|c} 0 & -1 & \multirow{2}{*}{O} \\ -1 & 0 & \\ \hline \multicolumn{2}{c|}{O} & A \end{array}\right), \quad \left(\begin{array}{cc|c} 0 & 1 & \multirow{2}{*}{O} \\ 1 & 0 & \\ \hline \multicolumn{2}{c|}{O} & A \end{array}\right)$$

As to the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, by applying the operation II, by adding the third row to the first row and the third column to the first column and by subtracting the second row from the third row and the second column from the third column, we obtain a sequence of matrices

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

By applying the inverse of the operation II and by subtracting the first row from the row and the first column from the second column, we obtain

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Therefore, regardless of being b'_{00} odd or even, it follows that $\mathrm{rank}B = \mathrm{rank}A + 2$, $\#W' = \#W + 2$ and $\mathrm{sign}B = \mathrm{sign}A$.

Now we consider the case that the number of components of $D(L)$ is altered. In this case, note that W_0 and W_0' are the same region. Therefore the Goeritz matrices A and B can be expressed as follows:

$$A = \left(\begin{array}{c|c} A_1 & O \\ \hline O & \widetilde{A}_2 \end{array}\right), \quad B = \left(\begin{array}{c|c|c} A_1 & O & O \\ \hline O & 0 & O \\ \hline O & O & A_2 \end{array}\right)$$

Here, A_1 is the Goeritz matrix of some link diagram. Also, $\widetilde{A}_2$ is the $\widetilde{G}$ matrix for an appropriate link diagram and A_2 is the Goeritz matrix obtained from $\widetilde{A}_2$ by deleting the first row and the first column. Hence, by using the operation I several times A can be transformed to B, which yields that rank $B = \mathrm{rank}A$, $\#W' = \#W + 2$ and $\mathrm{sign}B = \mathrm{sign}A$.

(4) Type IIIa (Case 1)

In this case, we have

$$\widetilde{A} = \begin{pmatrix} a_{11} & a_{12}-1 & a_{13}+1 & \cdots \\ a_{21}-1 & a_{22}+1 & a_{23}-1 & \cdots \\ a_{31}+1 & a_{32}-1 & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}, \quad \widetilde{B} = \begin{pmatrix} a_{11}-1 & a_{12} & a_{13} & 1 & \cdots \\ a_{21} & a_{22} & a_{23} & -1 & \cdots \\ a_{31} & a_{32} & a_{33}-1 & 1 & \cdots \\ 1 & -1 & 1 & -1 & 0 \\ \vdots & \vdots & \vdots & 0 & \vdots \end{pmatrix}$$

In $\widetilde{A}$ and $\widetilde{B}$, the first row and the first column correspond to W_1, the second row and the second column correspond to W_2 and the third row and the third column correspond to W_3. Furthermore, the fourth row and the fourth column of $\widetilde{B}$ correspond to W_4. Hence, the Goeritz matrices obtained by deleting the first row and the first column will be as follows:

$$A = \begin{pmatrix} a_{22}+1 & a_{23}-1 & \cdots \\ a_{32}-1 & a_{33} & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix}, \quad B = \begin{pmatrix} a_{22} & a_{23} & -1 & \cdots \\ a_{32} & a_{33}-1 & 1 & \cdots \\ -1 & 1 & -1 & 0 \\ \vdots & \vdots & 0 & \vdots \end{pmatrix}$$

In the matrix B, by subtracting the third row from the first row and the third column from the first column and by adding the third row to the second row and the third column to the second column, we obtain

$$\begin{pmatrix} a_{22}+1 & a_{23}-1 & 0 & \cdots \\ a_{32}-1 & a_{33} & 0 & \cdots \\ 0 & 0 & -1 & 0 \\ \vdots & \vdots & 0 & \vdots \end{pmatrix},$$

which will be transformed to A by the operation I and the inverse of the operation II and we can conclude that $\operatorname{rank}B=\operatorname{rank}A+1$, $\#W'=\#W+1$ and $\operatorname{sign}(B)=\operatorname{sign}(A)-1$.

(5) Type IIIa (Case 2)

In this case, we have

$$\widetilde{A}=\begin{pmatrix} a_{11} & a_{12} & a_{13} & 1 & \cdots \\ a_{21} & a_{22}-1 & a_{23} & -1 & \cdots \\ a_{31} & a_{32} & a_{33}-1 & 1 & \cdots \\ 1 & -1 & 1 & -1 & 0 \\ \vdots & \vdots & \vdots & 0 & \vdots \end{pmatrix}, \quad \widetilde{B}=\begin{pmatrix} a_{11}-1 & a_{12}-1 & a_{13}+1 & \cdots \\ a_{21}-1 & a_{22} & a_{23}-1 & \cdots \\ a_{31}+1 & a_{32}-1 & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

In $\widetilde{A}$ and $\widetilde{B}$, the first row and the first column correspond to W_1, the second row and the second column correspond to W_2 and the third row and the third column correspond to W_3. Moreover, the fourth row and the fourth column of $\widetilde{A}$ correspond to W_4. Hence, the Goeritz matrices obtained by deleting the first row and the first column will be as follows:

$$A=\begin{pmatrix} a_{22}-1 & a_{23} & -1 & \cdots \\ a_{32} & a_{33}-1 & 1 & \cdots \\ -1 & 1 & -1 & 0 \\ \vdots & \vdots & 0 & \vdots \end{pmatrix}, \quad B=\begin{pmatrix} a_{22} & a_{23}-1 & \cdots \\ a_{32}-1 & a_{33} & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix}$$

In the matrix A, we subtract the third row from the first row and the third column from the first column, and add the second row to the third row and the second column to the third column. Then we obtain the following matrix:

$$
\begin{pmatrix}
a_{22} & a_{23}-1 & 0 & \cdots \\
a_{32}-1 & a_{33} & 0 & \cdots \\
0 & 0 & -1 & 0 \\
\vdots & \vdots & 0 & \vdots
\end{pmatrix},
$$

which can be transformed to B by the operation I and the inverse of the operation II. Hence it follows that $\mathrm{rank}B=\mathrm{rank}A-1$, $\#W'=\#W-1$ and $\mathrm{sign}B=\mathrm{sign}A+1$.

(6) Type IIIb (Case 1)

In this case, we have

$$
\widetilde{A}=\begin{pmatrix}
a_{11} & a_{12}+1 & a_{13}+1 & \cdots \\
a_{21}+1 & a_{22} & a_{23}-1 & \cdots \\
a_{31}+1 & a_{32}-1 & a_{33} & \cdots \\
\vdots & \vdots & \vdots & \vdots
\end{pmatrix},\quad
\widetilde{B}=\begin{pmatrix}
a_{11}-1 & a_{12} & a_{13} & 1 & \cdots \\
a_{21} & a_{22}+1 & a_{23} & -1 & \cdots \\
a_{31} & a_{32} & a_{33}+1 & -1 & \cdots \\
1 & -1 & -1 & 1 & 0 \\
\vdots & \vdots & \vdots & 0 & \vdots
\end{pmatrix}
$$

In $\widetilde{A}$ and $\widetilde{B}$, the first row and the first column correspond to W_1, the second row and the second column correspond to W_2 and the third row and the third column correspond to W_3. In the matrix $\widetilde{B}$, the fourth row and the fourth column correspond to W_4. Hence, the Goeritz matrices obtained by deleting the first row and the first column will be as follows:

$$
A=\begin{pmatrix}
a_{22} & a_{23}-1 & \cdots \\
a_{32}-1 & a_{33} & \cdots \\
\vdots & \vdots & \vdots
\end{pmatrix},\quad
B=\begin{pmatrix}
a_{22}+1 & a_{23} & -1 & \cdots \\
a_{32} & a_{33}+1 & 1 & \cdots \\
-1 & -1 & 0 & 0 \\
\vdots & \vdots & & \vdots
\end{pmatrix}
$$

In the matrix B, by adding the third row to the first row and the third column to the first column and by adding the third row to the second row and the third column to the second column, we obtain

$$
\begin{pmatrix}
a_{22} & a_{23}-1 & 0 & \cdots \\
a_{32}-1 & a_{33} & 0 & \cdots \\
0 & 0 & 1 & 0 \\
\vdots & \vdots & 0 & \vdots
\end{pmatrix},
$$

which will be transformed to A by the operation I and the inverse of the operation II. Therefore we have $\mathrm{rank}B=\mathrm{rank}A+1$, $\#W'=\#W+1$ and $\mathrm{sign}B=\mathrm{sign}A+1$

(7) Type IIIb (Case 2)

$$\widetilde{A}=\begin{pmatrix} a_{11}+1 & a_{12} & a_{13} & -1 & \cdots \\ a_{21} & a_{22}+1 & a_{23} & -1 & \cdots \\ a_{31} & a_{32} & a_{33} & 1 & \cdots \\ -1 & -1 & 1 & 1 & 0 \\ \vdots & \vdots & \vdots & 0 & \vdots \end{pmatrix}, \quad \widetilde{B}=\begin{pmatrix} a_{11} & a_{12}-1 & a_{13}+1 & \cdots \\ a_{21}-1 & a_{22} & a_{23}+1 & \cdots \\ a_{31}+1 & a_{32}+1 & a_{33}-1 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

In $\widetilde{A}$ and $\widetilde{B}$, the first row and the first column correspond to W_1, the second row and the second column correspond to W_2 and the third row and the third column correspond to W_3. In $\widetilde{A}$, the fourth row and the fourth column correspond to W_4. Hence, the Goeritz matrices obtained by deleting the first row and the first column will be as follows:

$$A=\begin{pmatrix} a_{22}+1 & a_{23} & -1 & \cdots \\ a_{32} & a_{33} & 1 & \cdots \\ -1 & 1 & 1 & 0 \\ \vdots & \vdots & 0 & \vdots \end{pmatrix}, \quad B=\begin{pmatrix} a_{22} & a_{23}+1 & \cdots \\ a_{32}+1 & a_{33}-1 & \cdots \\ \vdots & \vdots & \vdots \end{pmatrix}$$

In the matrix A, we add the third row to the first row and the third column to the first column, and we subtract the third row from the second row and the third column from the second column. Then we obtain the following matrix:

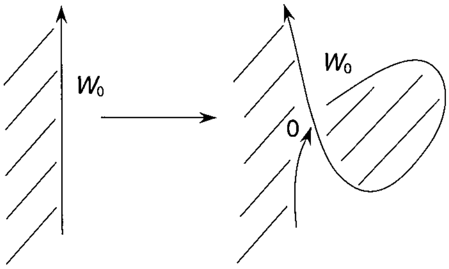
$$\begin{pmatrix} a_{22} & a_{23}+1 & 0 & \cdots \\ a_{32}+1 & a_{33}-1 & 0 & \cdots \\ 0 & 0 & 1 & 0 \\ \vdots & \vdots & 0 & \vdots \end{pmatrix},$$

which will be transformed to the matrix B by the operation I and the inverse of the operation II. This shows that $\mathrm{rank}B=\mathrm{rank}A-1$, $\#W'=\#W-1$ and $\mathrm{sign}B=\mathrm{sign}A-1$.

§ 3. $\mu(D(L))$ and the Reidemeister Moves

In this section, we study the link diagrams $D(L)$ of oriented links L and investigate the effect on $\mu(D(L))$ under the Reidemeister Moves. We need to consider the following 40 cases shown in the figures from 3.1 to 3.40. In each case, $\Delta\mu$ denotes the difference of $\mu(D(L))$ under the operation given in each figure:

Type Ia (Case 1)



$\Delta\,\mu = 0$ (No crossings of Type II)

Fig. 3. 1

Type Ia (Case 2)

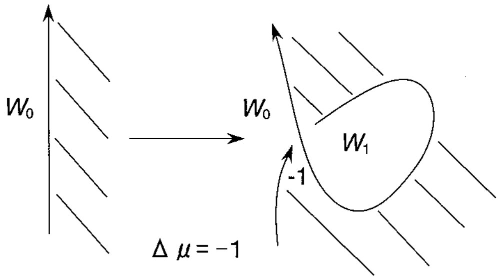
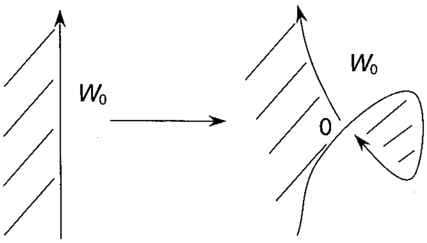


Fig. 3. 2

Type Ib (Case 1)



$\Delta\,\mu = 0$ (No crossings of Type II)

Fig. 3. 3

Type Ib (Case 2)

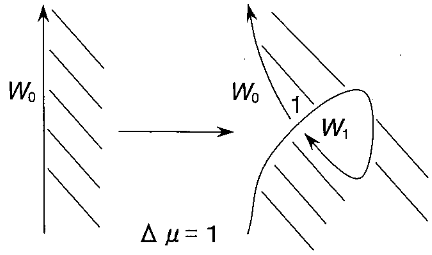
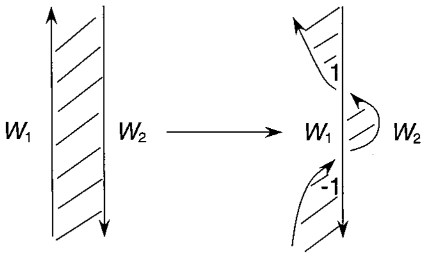


Fig. 3. 4

Type II (Case 1-1)



$\Delta\,\mu = 0$ (No crossings of Type II)

Fig. 3. 5

Type II (Case 1-2)

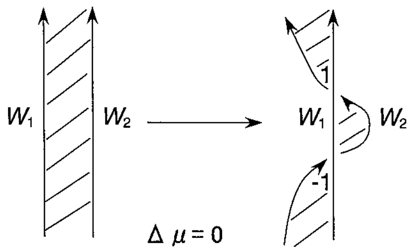


Fig. 3. 6

Type II (Case 2 −1)

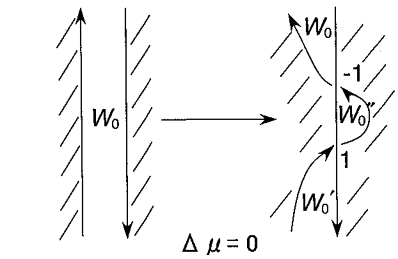


Fig. 3. 7

Type II (Case 2 −2)

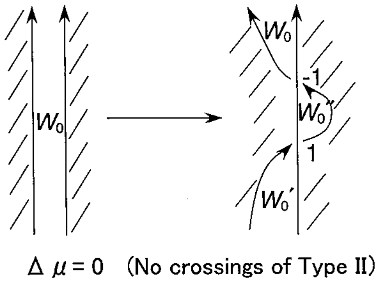


Fig. 3. 8

Type IIIa (Case 1−1)

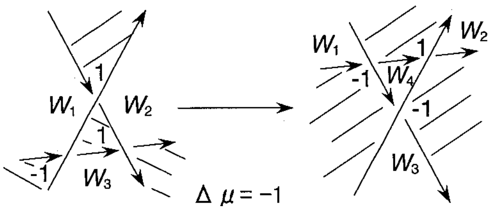


Fig. 3. 9

Type IIIa (Case 1−2)

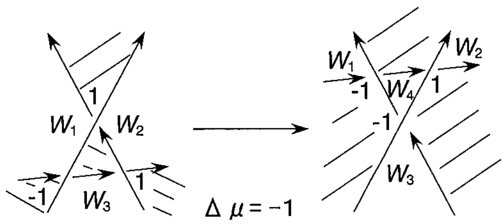


Fig. 3. 10

Type IIIa (Case 1−3)

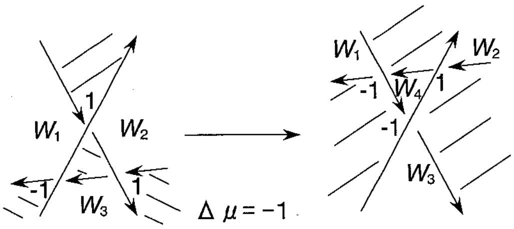


Fig. 3. 11

Type IIIa (Case 1−4)

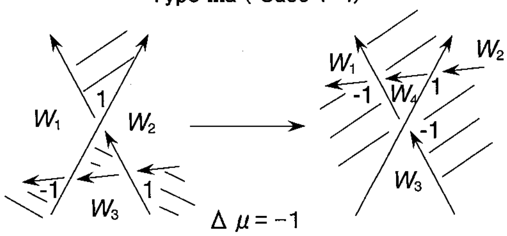


Fig. 3. 12

Type IIIa (Case 1−5)

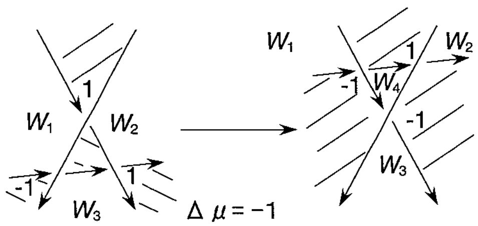


Fig. 3. 13

Type IIIa (Case 1−6)

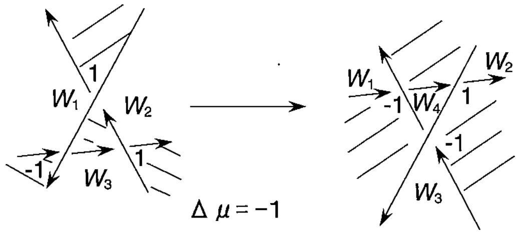


Fig. 3. 14

Type IIIa (Case 1–7)

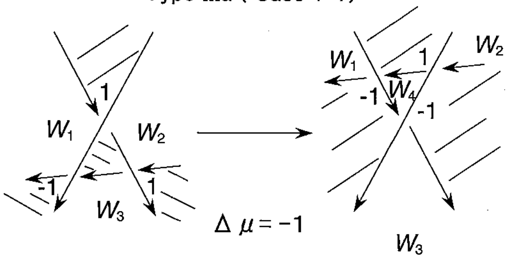


Fig. 3. 15

Type IIIa (Case 1–8)

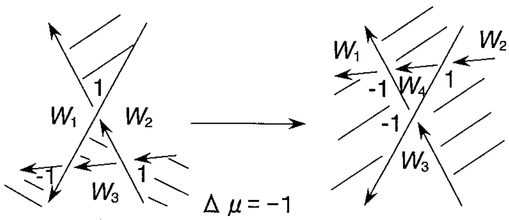


Fig. 3. 16

Type IIIa (Case 2–1)

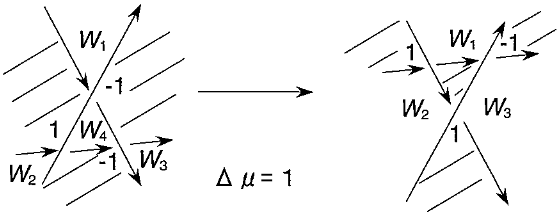


Fig. 3. 17

Type IIIa (Case 2–2)

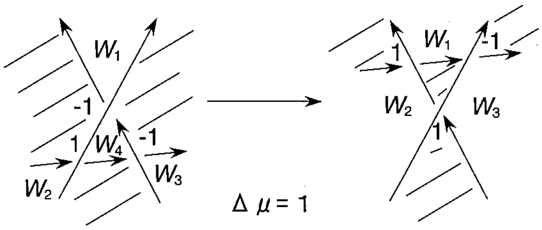


Fig. 3. 18

Type IIIa (Case 2–3)

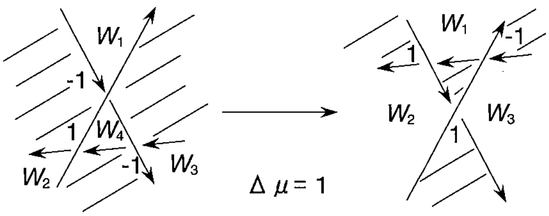


Fig. 3. 19

Type IIIa (Case 2–4)

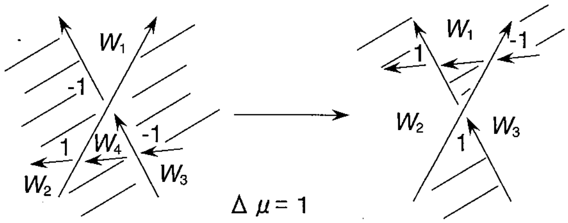


Fig. 3. 20

Type IIIa (Case 2–5)

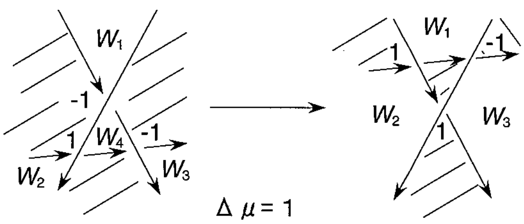


Fig. 3. 21

Type IIIa (Case 2–6)

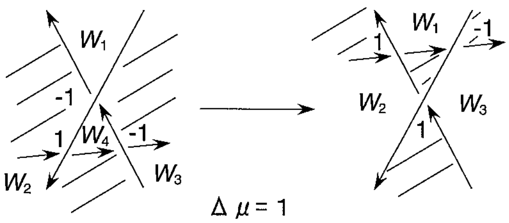


Fig. 3. 22

Type IIIa (Case 2–7)

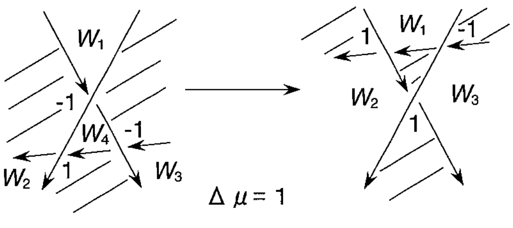


Fig. 3. 23

Type IIIa (Case 2–8)

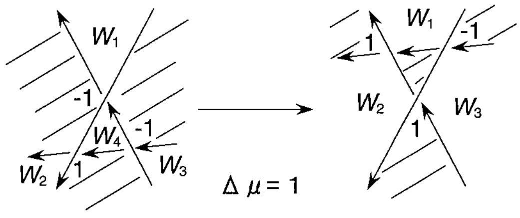


Fig. 3. 24

Type IIIb (Case 1–1)

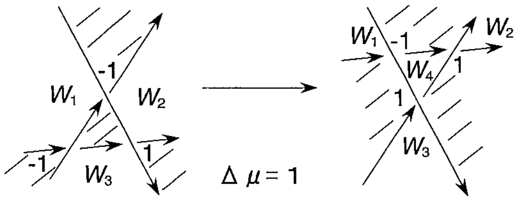


Fig. 3. 25

Type IIIb (Case 1–2)

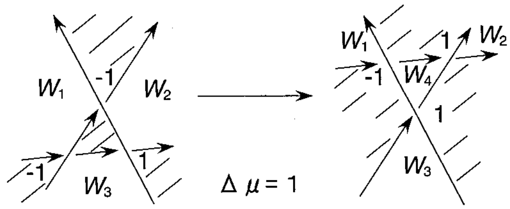


Fig. 3. 26

Type IIIb (Case 1–3)

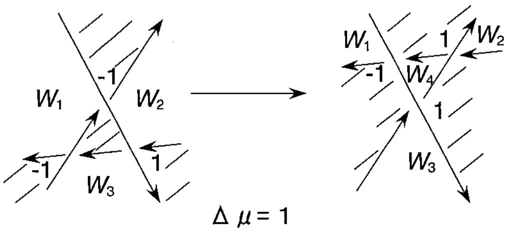


Fig. 3. 27

Type IIIb (Case 1–4)

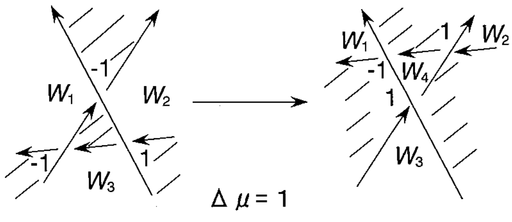


Fig. 3. 28

Type IIIb (Case 1–5)

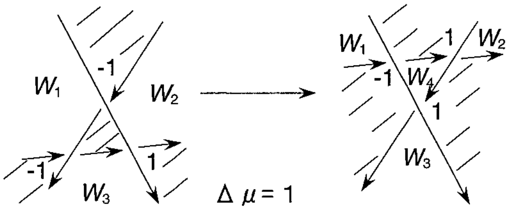


Fig. 3. 29

Type IIIb (Case 1–6)

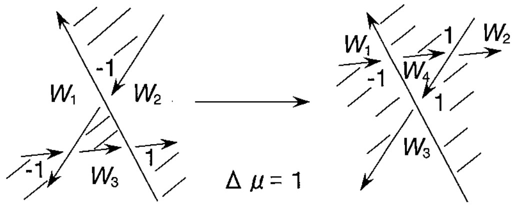


Fig. 3. 30

Type IIIb (Case 1–7)

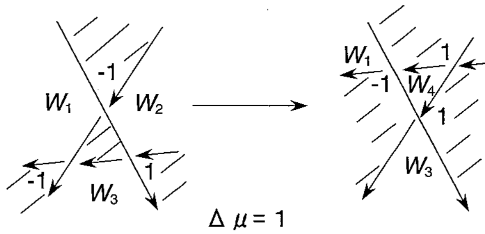


Fig. 3. 31

Type IIIb (Case 1–8)

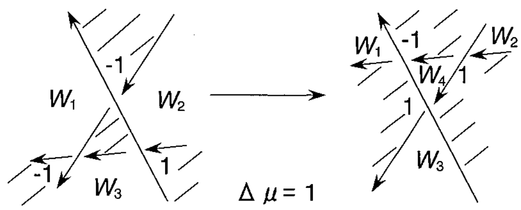


Fig. 3. 32

Type IIIb (Case 2–1)

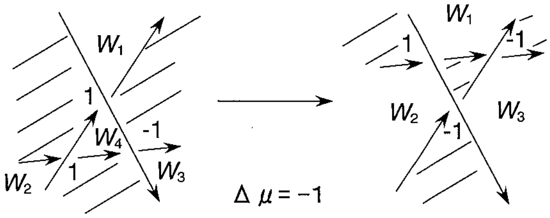


Fig. 3. 33

Type IIIb (Case 2–2)

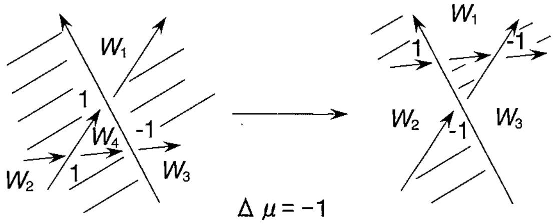


Fig. 3. 34

Type IIIb (Case 2–3)

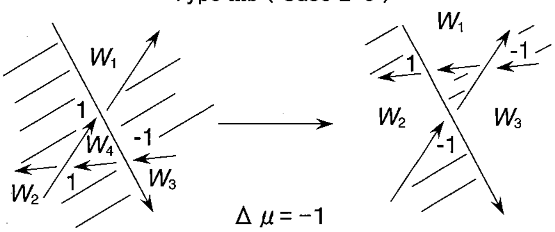


Fig. 3. 35

Type IIIb (Case 2–4)

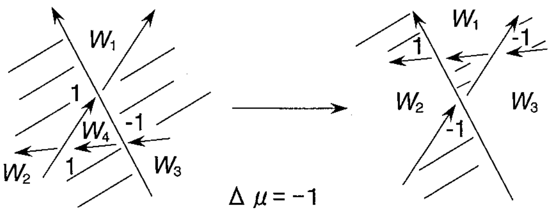


Fig. 3. 36

Type IIIb (Case 2–5)

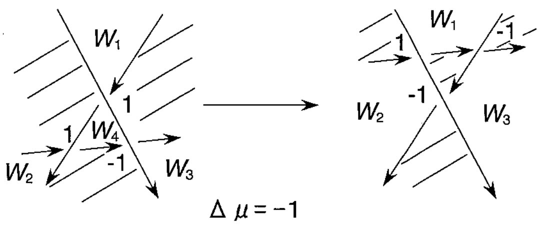


Fig. 3. 37

Type IIIb (Case 2–6)

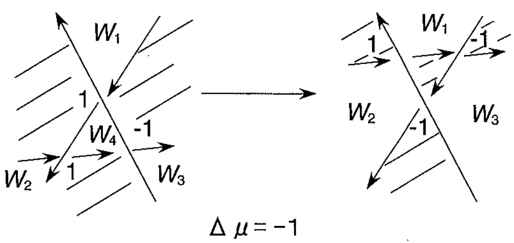


Fig. 3. 38

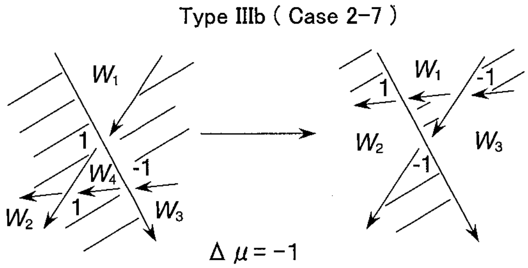


Fig. 3. 39

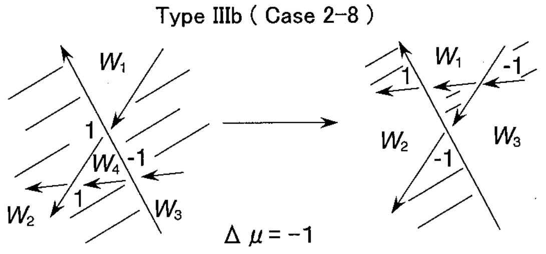


Fig. 3. 40

§ 4. Proofs for the theorems

We summarize the arguments in §2 and §3 in the following table, where ΔW, Δrank, Δsign and $\Delta\mu$ denote the difference of $\#W(D(L))$, $\mu(D(L))$, rank$G(L)$ and sign$G(L)$ under the operation given in each figure from Fig. 2. 1 to Fig. 2. 10. This table shows that Theorem 1, Theorem 2 and Theorem 3 hold in each of the Reidemeister Moves, from which we obtain our theorems. Note that since we have rank$G(L) \equiv$ sign$G(L)$ (mod 2), we can also show that Theorem 3 implies Theorem 2.

	ΔW	Δrank	Δsign	$\Delta\mu$
Type Ia(Case 1)	0	0	0	0
Type Ia(Case 2)	1	1	-1	-1
Type Ib(Case 1)	0	0	0	0
Type Ib(Case 2)	1	1	1	1
Type II(Case 1)	0	0	0	0
Type II(Case 2)	2	2	0	0
	2	0	0	0
Type IIIa(Case 1)	1	1	-1	-1
Type IIIa(Case 2)	-1	-1	1	1
Type IIIb(Case 1)	1	1	1	1
Type IIIb(Case 2)	-1	-1	-1	-1

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